Engineering Notes

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Development of Aircraft Vortex Wakes in Turbulent Flow

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Nomenclature

$\bar{c}$ = mean wing chord
f = functional dependence of Reynolds stresses on λ
r = radial distance from vortex center
R = radius of vortex core
t = time
u = radial velocity
V = peak tangential velocity
w = axial velocity
Z = distance along vortex axis
ζ = Z component of vorticity
λ = nondimensional radius = r/R
ν = kinematic viscosity
ξ = r component of vorticity
ϕ = functional dependence of ζ on λ
Ω, = vorticity of core

Superscripts

$(\ )'$ = fluctuating component, or differentiation with respect to λ
$\overline{(\ )}$ = time averaged quantity

Introduction

TRAILING vortices generated at the tips of finite lifting surfaces induce velocities that can persist for considerable lengths of time. The peak velocities in such vortices represent a very real danger to light aircraft. The prediction of the time-dependent characteristics of these vortices is, therefore, an important problem.

A method for predicting the time dependence of vortex size, strength, and peak tangential velocity in turbulent flow is considered in this Note. The existence of a similarity solution is assumed for large times and the turbulent shearing stresses are represented in general functional form without recourse to mixing length or eddy viscosity assumptions. Although the similarity solution is not obtained (as indeed it cannot be without the assumption of a specific form for the turbulence shearing stress terms), the time dependency of the core radius, vorticity and peak tangential velocity can nevertheless be established. The validity of the method will be indicated by comparison with empirical data.

Analytical Considerations

The governing equation is obtained from the Z component of the Navier-Stokes equation written in terms of vorticity. Under the assumptions of axial symmetry and full development in the Z direction, this equation, Reynolds averaged to account for turbulence, can be written as

$$\frac{\partial \zeta}{\partial t} + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)(\overline{u'\zeta'} - \overline{w'\xi'}) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\zeta \quad (1)$$

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Squire[1] evaluated Eq. (1) assuming an eddy viscosity form for the Reynolds stress term. The eddy viscosity was taken to be proportional to the vortex strength and the resulting equation was solved in a manner analogous to the laminar case.

Rather than considering eddy viscosity, consider a more general approach which does not require specification of the form of the turbulent shearing stress terms. Far from solid boundaries these shearing stresses are considerably larger than molecular shearing stresses for certain types of flows (e.g., wakes, jets, outer region of turbulent boundary layers). Applying this observation to Eq. (1), the right-hand terms can be neglected in comparison to those on the left.

Assuming that a similarity solution exists for large values of t, the dependent variables can be written as a product of two functions in the manner of the free parameter method described by Hansen.[2] The first of these functions is taken to be dependent upon t alone while the second is a function of both t and r. The time-dependent functions are taken to be dimensionally appropriate combinations of the scaling parameters R and Ω, while the remaining functions are taken to depend upon the nondimensional radius $\lambda = r/R$. The resulting forms of the dependent variables are

$$\zeta = \Omega(t)\phi(\lambda) \qquad (\overline{u'\zeta'} - \overline{w'\xi'}) = R(t)\Omega^2(t)f(\lambda) \quad (2)$$

Using Eq. (2), the resulting form of Eq. (1) is

$$\left(\frac{1}{\Omega^2}\frac{d\Omega}{dt}\right)\phi - \left(\frac{1}{\Omega R}\frac{dR}{dt}\right)\lambda\phi' + \frac{f(\lambda)}{\lambda} = f'(\lambda) \quad (3)$$

where the prime indicates differentiation with respect to λ. If a similarity solution does in fact exist, the coefficients of the first two terms must be independent of time. This, of course, implies that $\Omega \sim 1/t$ and $R \sim t^k$ where k is a constant.

Measurements[3,4] indicate that the core circulation is nearly constant over extended periods of time. Thus, $R^2\Omega$ must be nearly constant and k is approximately $\frac{1}{2}$. Since the peak velocity at the edge of the vortex core must vary as the product $R\Omega$, it follows that $V \sim t^{-1/2}$. To summarize,

$$\Omega \sim (1/t) \qquad R \sim t^{1/2} \qquad V \sim t^{-1/2} \quad (4)$$

Comparison with Experiment

Measurements of trailing vortex characteristics downstream of a full-scale light aircraft have been presented by McCormick, Tangler, and Sherrieb.[4] The authors conclude that for large downstream distances (approximately $Z/\bar{c} > 250$) the peak tangential velocities vary as the inverse square root of the distance. Since the distance downstream is directly proportional to time, the latter of the relations given in Eq. (4) is substantiated empirically.

Also presented in Ref. 4 are measurements of core radius. Data scatter, however, prevents any conclusion being drawn as to the validity of the second of the relations in Eq. (4).

Conclusions

The method of solution proposed predicts the time variation of core radius, vorticity, and peak tangential velocities given in Eq. (4). These results can be valid only for times sufficiently greater than zero over which the core circulation is nearly constant. The decay of peak tangential velocities given in Eq. (4) is substantiated by empirical results.

References

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A Closed-Form Solution to Oblique Shock-Wave Properties

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THIS Note is concerned with the direct computation of oblique shock-wave properties with freestream Mach number and flow-deflection angle as the independent variables.

The equations governing oblique shock relations may be written in the form

$$\sin^6\theta + b\sin^4\theta + c\sin^2\theta + d = 0$$

where

$$b = -\left[\frac{M_1^2 + 2}{M_1^2}\right] - \gamma\sin^2\delta$$

$$c = (2M_1^2 + 1)/M_1^4 + [(\gamma + 1)^2/4 + (\gamma - 1)/M_1^2]\sin^2\delta$$

$$d = -\cos^2\delta/M_1^4$$

and

θ = shock-wave angle
M_1 = freestream Mach number
δ = deflection angle
γ = ratio of specific heats

which is cubic in $\sin^2\theta$, having three real roots, the smallest of which results in a decrease in entropy.

Contrary to the statement of Ref. 1, that no convenient explicit relation exists for this case, there is indeed a general solution for a cubic. The mathematical derivation can be found in Ref. 2. From Ref. 2, the solution for a cubic having three real roots is

$$\sin^2\theta = -b/3 + \tfrac{2}{3}(b^2 - 3c)^{1/2}\cos[(\phi + n\pi)/3]$$

where

$$\cos\phi = (\tfrac{9}{2}bc - b^3 - \tfrac{27}{2}d)/(b^2 - 3c)^{3/2}$$

and $n = 0$ corresponds to the strong shock solution; $n = 2$ results in a decrease in entropy; and $n = 4$ corresponds to the weak shock solution. Although many readers may be aware of this solution, the wide use of iteration schemes to solve this problem has prompted the author to set down the explicit solution in general terms.

References

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Free Vibration of Simply Supported Parallelogrammic Plates

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Nomenclature

a,b = dimensions of the plate, see Fig. 1a
D = plate rigidity, $Eh^3/12(1 - \nu^2)$
E = Young's modulus of the material of the plate
h = plate thickness
k = frequency parameter, $(\rho h/D)^{1/2}\omega a^2/\pi^2$
k_m = frequency parameter of membrane, $(\mu/S)^{1/2}\omega a/\pi$
m,n = number of half sine waves in the two directions x_1 and y_1, respectively
S = uniform tension per unit length of stretched membrane
x,y = rectangular coordinate system defined in Fig. 1a
x_1,y_1 = oblique coordinates defined in Fig. 1a
ρ = mass density of the plate material
ψ = angle of skew, defined in Fig. 1a
ω = frequency of oscillation in rad/sec
μ = mass per unit area of membrane
ν = Poisson's ratio

Introduction

IN this Note, the results of numerical calculations for the first few frequencies of simply supported parallelogrammic plates, using the Rayleigh-Ritz method employing double Fourier sine series in oblique coordinates, are presented. Interesting features, hitherto unreported in the literature, such as 1) the skew angle splitting the degenerate frequencies of rectangular plates to distinct ones and 2) the "frequency crossing" of the modes of simply supported skew plates, are discussed. In fact, it has been shown in Ref. 1 that these features are also exhibited by clamped skew plates.

The literature does not contain adequate results for the frequencies of simply supported parallelogrammic plates. Conway and Farnham[2] calculated only the fundamental frequency for a few configurations of the plate by point matching, using a mathematical relationship that exists between the problems of a simply supported polygonal plate and a polygonal membrane of the same geometry.[3–5] This relationship shows that the eigenvalues of the plate are squares of the eigenvalues of the membrane, whereas the eigenfunctions are identical. Weinstein[6] reports the upper and lower bounds of the frequencies of modes symmetric about both the diagonals of rhombic membrane, which have been calculated by Stadter[7] in an unpublished report. These values serve admirably for comparison with the results of simply supported rhombic plates on the basis of the aforementioned relationship.

Details of Solution

The vibration problem becomes a particular case of panel-flutter problem of simply supported parallelogrammic panels, which is discussed in detail in Ref. 8. Consequently, the

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